# Limits — problems and solutions

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last updated: February 13, 2019

Summary: This document contains some of the most common limits problems for you to review! Feel free to jump around or start from the beginning! Visit *https://sciency.tech* for the solutions and other problem-and-solution guides!

## Contents

# 1 How to read limits out loud

1. How do you read $f(x)$?

**Solution:** "F" of "X."

2. How do you read $\lim_{x \to a} f(x) = L$?

**Solution:** The limit of "F" as "X" approaches "A" is "L."

3. How do you read $\lim_{x \to a^-} f(x)$?

**Solution:** The limit of "F" as "X" approaches "A" from the left.

4. How do you read $\lim_{x \to a^+} f(x)$?

**Solution:** The limit of "F" as "X" approaches "A" from the right.

# 2 Basic limit problems

1. $\lim\limits_{x \to 3} x = ?$

> **Solution:**
> $$\lim_{x \to 3} x = 3.$$

2. $\lim\limits_{x \to a} (x^2 + 7) = ?$

> **Solution:**
> $$\lim_{x \to a} \left(x^2 + 7\right) = a^2 + 7.$$

3. $\lim\limits_{x \to \pi} \cos\left(\frac{x}{2}\right) = ?$

> **Solution:**
> $$\begin{aligned} \lim_{x \to \pi} \cos\left(\frac{x}{2}\right) &= \cos\left(\frac{\pi}{2}\right) \\ &= 1. \end{aligned}$$

4. $\lim\limits_{x \to \infty} e^{-x} = ?$

> **Solution:**
> $$\begin{aligned} \lim_{x \to \infty} e^{-x} &= e^{-\infty} \\ &= 1. \end{aligned}$$

5. $\lim\limits_{x \to a} \dfrac{x-3}{x^2+7} = ?$

**Solution:**

$$\lim_{x \to a} \frac{x-3}{x^2+7} = \frac{a-3}{a^2+7}.$$

6. $\lim\limits_{x \to \pi} x \cos x = ?$

**Solution:**

$$\begin{aligned} \lim_{x \to \pi} x \cos x &= \pi \cos \pi \\ &= -\pi. \end{aligned}$$

# 3 One-sided limits

1. Let

$$f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ 3x-7, & \text{if } x \geq 0 \end{cases},$$

then

$$\lim_{x \to 0^+} f(x) = ?$$

**Solution:**

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (3x-7) \\ &= 0-7 \\ &= 7. \end{aligned}$$

2. Let

$$f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ 3x-7, & \text{if } x \geq 0 \end{cases}$$

then

$$\lim_{x \to 0^-} f(x) = ?$$

**Solution:**

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (x+2) \\ &= 0+2 \\ &= 2. \end{aligned}$$

3. Let

$$f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ 3x-7, & \text{if } x \geq 0 \end{cases},$$

then

$$\lim_{x\to 0} f(x) = ?$$

**Solution:**

$$\lim_{x\to 0} f(x) \text{ does not exist}$$

since

$$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x).$$

Recall from the previous two questions that

$$\lim_{x\to 0^-} f(x) = 2$$

and

$$\lim_{x\to 0^+} f(x) = -7.$$

# 4 Limit laws

1. $\lim_{x \to a} (f(x) + g(x)) = ?$

**Solution:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

2. $\lim_{x \to a} f(x)\ g(x) = ?$

**Solution:**

$$\lim_{x \to a} (f(x)\ g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right).$$

3. $\lim_{x \to a} \frac{f(x)}{g(x)} = ?$

**Solution:**

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

4. $\lim_{x \to a} f(g(x)) = ?$

**Solution:**

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right),$$

assuming $f$ is a continuous function.

5. $\lim_{x \to a} 17 = ?$

**Solution:**

$$\lim_{x \to a} 17 = 17.$$

6. $\lim_{x \to a} \left(f(x)\right)^2 = ?$

**Solution:**
$$\lim_{x \to a} \left(f(x)\right)^2 = \left(\lim_{x \to a} f(x)\right)^2.$$

7. $\lim_{x \to a} \left(f(x)\right)^n = ?$

**Solution:**
$$\lim_{x \to a} \left(f(x)\right)^n = \left(\lim_{x \to a} f(x)\right)^n.$$

8. $\lim_{x \to a} (7x-2)^3 = ?$

**Solution:**
$$\begin{aligned} \lim_{x \to a} (7x-2)^3 &= \left(\lim_{x \to a} (7x-2)\right)^3 \\ &= (7a-2)^3. \end{aligned}$$

9. $\lim_{x \to 0} \sqrt{x+4} = ?$

**Solution:** Since the square root function is continuous,
$$\begin{aligned} \lim_{x \to 0} \sqrt{x+4} &= \sqrt{\lim_{x \to 0} (x+4)} \\ &= \sqrt{4} \\ &= 2. \end{aligned}$$

10. $\lim_{x \to -7} \sqrt{x+4} = ?$

**Solution:**

$$\begin{aligned} \lim_{x \to -7} \sqrt{x+4} &= \sqrt{-7+4} \\ &= \sqrt{-3} \\ &= i\sqrt{3}, \end{aligned}$$

where $i = \sqrt{-1}$ is called the "imaginary number."

# 5 Harder limit problems

1. $\lim\limits_{x\to 5} \dfrac{x^2-25}{x-5} = ?$

**Solution:**

$$\begin{aligned}\lim_{x\to 5} \frac{x^2-25}{x-5} &= \lim_{x\to 5} \frac{(x+5)(x-5)}{(x-5)} \\ &= \lim_{x\to 5}(x+5) \\ &= 5+5 \\ &= 10.\end{aligned}$$

2. $\lim\limits_{x\to\infty} \dfrac{1}{x} = ?$

**Solution:**

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x} &= \frac{1}{\lim\limits_{x\to\infty} x} \\ &= \frac{1}{\infty} \\ &= 0.\end{aligned}$$

3. $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = ?$

**Solution:**

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x^2} &= \frac{1}{\lim\limits_{x\to\infty} x^2} \\ &= \frac{1}{\infty} \\ &= 0.\end{aligned}$$

4. Let $a_n = 2 + 1/n$. Then $\lim_{n\to\infty} a_n = ?$

**Solution:**

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(2+\frac{1}{n}\right)\\ &= \lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n}\\ &= 2+0\\ &= 2.\end{aligned}$$

5. $\lim_{x\to 0} \dfrac{\sqrt{x^2+49}-7}{x^2} = ?$

**Solution:** First, notice that directly plugging 0 into $x$ gives us $\frac{0}{0}$, so we need another way.

If we instead multiply the top and bottom by

$$\sqrt{x^2+49}+7,$$

then we get

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x^2+49}-7}{x^2} &\cdot \frac{\sqrt{x^2+49}+7}{\sqrt{x^2+49}+7}\\ &= \lim_{x\to 0}\frac{(x^2+49)-49}{x^2\left(\sqrt{x^2+49}+7\right)}\\ &= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+49}+7\right)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+49}+7}\\ &= \frac{1}{7+7}\\ &= \frac{1}{14}.\end{aligned}$$

6. $\displaystyle \lim_{x\to 5} \frac{\sqrt{x^2+24}-7}{x^2-25} = ?$

**Solution:** Similar to the previous problem, directly plugging 5 into $x$ gives us $\frac{0}{0}$, so we need to use another approach. We try multiplying the top and bottom by $\sqrt{x^2+24}+7$:

$$\begin{aligned}
\lim_{x\to 5} \frac{\sqrt{x^2+24}-7}{x^2-25} \quad \cdot \quad & \frac{\sqrt{x^2+24}+7}{\sqrt{x^2+24}+7} \\
&= \lim_{x\to 5} \frac{(x^2+24)-49}{(x^2-25)(\sqrt{x^2+24}+7)} \\
&= \lim_{x\to 5} \frac{(x^2-25)}{(x^2-25)(\sqrt{x^2+24}+7)} \\
&= \lim_{x\to 5} \frac{1}{\sqrt{x^2+24}+7} \\
&= \frac{1}{\sqrt{25+24}+7} \\
&= \frac{1}{14}.
\end{aligned}$$

7. $\displaystyle \lim_{x\to\infty} \frac{\sqrt{4x^4+24x-7}}{x^2-25} = ?$
(Try calculating this limit **without** using l'Hôpital's rule.)

**Solution:** First, divide the numerator and denominator by $x^2$ and rewrite the limit in terms of $h = 1/x$. To do this, understand that letting $x \to \infty$ is the same as letting $h = \frac{1}{x} \to 0^+$ (see Note 2 for more details). This allows us to rewrite the limit as

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{4x^4+24x-7}}{x^2-25} \cdot \frac{1/x^2}{1/x^2} &= \lim_{h\to 0^+} \frac{\sqrt{4+24h^3-7h^4}}{1-25h^2} \\
&= \frac{\sqrt{4}}{1} \\
&= 2.
\end{aligned}$$

**Note 1:**

Since both the numerator and denominator approaches infinity, you may be tempted to use l'Hôpital's rule. Of course, you can and that is the standard approach, but it will be messy. Give it a try and decide for yourself which approach is better.

**Note 2:**
The new limit is for $h$ approaching zero from the right-hand side. This is because we require $h$ to remain positive ($h > 0$) at all times since $x$ remained positive as $x$ approached infinity. However, it actually doesn't matter here since both the numerator and denominator are continuous at $h = 0$.

8. $\lim_{x \to -\infty} \dfrac{5x^3 + 4x + 7}{25 - 2x^3} = ?$

(Try calculating this limit **without** using l'Hôpital's rule.)

**Solution:** First, divide both the numerator and denominator by $x^3$ and let $h = \dfrac{-1}{x}$. This allows us to rewrite the limit as

$$\begin{aligned} \lim_{x \to -\infty} \frac{5x^3 + 4x + 7}{25 - 2x^3} &= \lim_{h \to 0^+} \frac{5 + 4h^2 - 7h^3}{25h^3 - 2} \\ &= \frac{-5}{2}. \end{aligned}$$

# 6 l'Hôpital's rule

1. What is l'Hôpital's rule?

**Solution:** L'Hôpital's rule is a method that lets us use derivatives in evaluating limits involving "indeterminate forms," i.e. when a straight-forward approach gives us $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

More specificially, l'Hôpital's rule tells us that when

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty},$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

where the primes ( ' ) signify taking the derivative with respect to $x$.

2. $\lim_{x\to 0} \frac{\sin x}{x} = ?$

**Solution:** Since $\frac{\sin 0}{0} = \frac{0}{0}$, we apply l'Hôpital's rule:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{(\sin x)'}{(x)'} \\ &= \lim_{x\to 0} \frac{\cos x}{1} \\ &= \cos 0 \\ &= 1. \end{aligned}$$

3. $\lim\limits_{x\to -3} \dfrac{(x+3)^3}{x^2+9} = ?$

**Solution:** Since

$$\frac{\lim\limits_{x\to -3} (x+3)^3}{\lim\limits_{x\to -3} (x^2+9)} = \frac{0}{0},$$

we apply l'Hôpital's rule:

$$\begin{aligned} \lim_{x\to -3} \frac{(x+3)^3}{x^2+9} &= \lim_{x\to -3} \frac{3(x+3)^2}{2x} \\ &= \frac{0}{-6} \\ &= 0. \end{aligned}$$

4. $\lim\limits_{x\to \infty} \dfrac{e^x}{x^2+4} = ?$

**Solution:** Since

$$\frac{\lim\limits_{x\to \infty} e^x}{\lim\limits_{x\to \infty} (x^2+4)} = \frac{\infty}{\infty},$$

we apply l'Hôpital's rule, which gives us

$$\begin{aligned} \lim_{x\to \infty} \frac{e^x}{x^2+4} &= \lim_{x\to \infty} \frac{(e^x)'}{(x^2+4)'} \\ &= \lim_{x\to \infty} \frac{e^x}{2x}. \end{aligned}$$

Since

$$\frac{\lim\limits_{x\to \infty} e^x}{\lim\limits_{x\to \infty} 2x} = \frac{\infty}{\infty},$$

we apply l'Hôpital's rule a second time:

$$\begin{aligned} \lim_{x\to \infty} \frac{e^x}{2x} &= \lim_{x\to \infty} \frac{(e^x)'}{(2x)'} \\ &= \lim_{x\to \infty} \frac{e^x}{2} \\ &= \frac{\infty}{2} \\ &= \infty. \end{aligned}$$

And so, the answer is

$$\lim_{x\to \infty} \frac{e^x}{x^2+4} = \infty.$$