Limits — Harder limit problems

brought to you by *sciency.tech*

last updated: February 13, 2019

Summary: This document provides you a few harder limit problems and their solutions
Visit *https://sciency.tech* for the solutions and other problem-and-solution guides!

Harder limit problems

1. $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5} = ?$

Solution:

$$\begin{aligned} \lim_{x \to 5} \frac{x^2 - 25}{x - 5} &= \lim_{x \to 5} \frac{(x+5)(x-5)}{(x-5)} \\ &= \lim_{x \to 5} (x+5) \\ &= 5 + 5 \\ &= 10. \end{aligned}$$

2. $\lim\limits_{x \to \infty} \dfrac{1}{x} = ?$

Solution:

$$\begin{aligned} \lim_{x \to \infty} \frac{1}{x} &= \frac{1}{\lim\limits_{x \to \infty} x} \\ &= \frac{1}{\infty} \\ &= 0. \end{aligned}$$

3. $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = ?$

Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x^2} &= \frac{1}{\lim\limits_{x\to\infty} x^2} \\ &= \frac{1}{\infty} \\ &= 0.\end{aligned}$$

4. Let $a_n = 2 + 1/n$. Then $\lim\limits_{n\to\infty} a_n = ?$

Solution:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(2 + \frac{1}{n}\right) \\ &= \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} \\ &= 2 + 0 \\ &= 2.\end{aligned}$$

5. $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+49} - 7}{x^2} = ?$

Solution: First, notice that directly plugging 0 into x gives us $\dfrac{0}{0}$, so we need another way.

If we instead multiply the top and bottom by

$$\sqrt{x^2+49} + 7,$$

then we get

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+49}-7}{x^2} \quad &\cdot \quad \frac{\sqrt{x^2+49}+7}{\sqrt{x^2+49}+7} \\
&= \lim_{x\to 0} \frac{(x^2+49)-49}{x^2\left(\sqrt{x^2+49}+7\right)} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+49}+7\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+49}+7} \\
&= \frac{1}{7+7} \\
&= \frac{1}{14}.
\end{aligned}$$

6. $\lim\limits_{x\to 5} \dfrac{\sqrt{x^2+24}-7}{x^2-25} = ?$

Solution: Similar to the previous problem, directly plugging 5 into x gives us $\frac{0}{0}$, so we need to use another approach. We try multiplying the top and bottom by $\sqrt{x^2+24}+7$:

$$\begin{aligned}
\lim_{x\to 5} \frac{\sqrt{x^2+24}-7}{x^2-25} \quad &\cdot \quad \frac{\sqrt{x^2+24}+7}{\sqrt{x^2+24}+7} \\
&= \lim_{x\to 5} \frac{(x^2+24)-49}{(x^2-25)(\sqrt{x^2+24}+7)} \\
&= \lim_{x\to 5} \frac{(x^2-25)}{(x^2-25)(\sqrt{x^2+24}+7)} \\
&= \lim_{x\to 5} \frac{1}{\sqrt{x^2+24}+7} \\
&= \frac{1}{\sqrt{25+24}+7} \\
&= \frac{1}{14}.
\end{aligned}$$

7. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{4x^4+24x-7}}{x^2-25} = ?$
(Try calculating this limit **without** using l'Hôpital's rule.)

Solution: First, divide the numerator and denominator by x^2 and rewrite the limit in terms of $h = 1/x$. To do this, understand that letting $x \to \infty$ is the same as letting $h = \dfrac{1}{x} \to 0^+$ (see Note 2 for more details). This allows us to rewrite the limit as

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{4x^4+24x-7}}{x^2-25} \cdot \frac{1/x^2}{1/x^2} &= \lim_{h\to 0^+} \frac{\sqrt{4+24h^3-7h^4}}{1-25h^2} \\ &= \frac{\sqrt{4}}{1} \\ &= 2.\end{aligned}$$

Note 1:
Since both the numerator and denominator approaches infinity, you may be tempted to use l'Hôpital's rule. Of course, you can and that is the standard approach, but it will be messy. Give it a try and decide for yourself which approach is better.

Note 2:
The new limit is for h approaching zero from the right-hand side. This is because we require h to remain positive ($h > 0$) at all times since x remained positive as x approached infinity. However, it actually doesn't matter here since both the numerator and denominator are continuous at $h = 0$.

8. $\displaystyle\lim_{x\to-\infty} \frac{5x^3+4x+7}{25-2x^3} = ?$

(Try calculating this limit **without** using l'Hôpital's rule.)

Solution: First, divide both the numerator and denominator by x^3 and let $h = \dfrac{-1}{x}$. This allows us to rewrite the limit as

$$\begin{aligned}\lim_{x\to-\infty} \frac{5x^3+4x+7}{25-2x^3} &= \lim_{h\to 0^+} \frac{5+4h^2-7h^3}{25h^3-2} \\ &= \frac{-5}{2}.\end{aligned}$$